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Reducibility of Types in Typed Lambda Calculus*

Comment on a Paper by Richard Statman

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Consider types built up from a base type 0 using the operation $\to$. A type σ is reducible to a type τ, notation $\sigma \leqslant \tau$, iff there exists a closed term M in $\sigma \to \tau$ such that for all closed N_1, N_2 in σ we have $N_1 =_{\beta\eta} N_2 \Leftrightarrow MN_1 =_{\beta\eta} MN_2$. Two types are equivalent iff each is reducible to the other. In (Statman, 1980, *in* "To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism" (J. P. Seldin and J. R. Hindley, Eds.), pp. 511–534, Academic Press, New York/London) is shown that the equivalence classes of types are well ordered in type $\omega + 2$ or $\omega + 3$. The paper does not decide if it is $\omega + 2$ or $\omega + 3$ because it is not clear whether $\mu \equiv (((0 \to 0) \to 0) \to 0) \to 0 \to 0$ and $\nu \equiv (0 \to 0) \to (0 \to 0) \to 0 \to 0$ are equivalent. We show that μ and ν are not equivalent and conclude that the equivalence classes are ordered in type $\omega + 3$.

1. INTRODUCTION

DEFINITION-NOTATION 1. Type, the set of types, is inductively defined as follows: (1) $0 \in$ Type; (2) $\sigma, \tau \in \text{Type} \Rightarrow \sigma \to \tau \in \text{Type}$. Λ^τ is the set of all typed λ-terms.

A type σ is *reducible* to a type τ, notation $\sigma \leqslant \tau$, iff there exists a closed term M in $\sigma \to \tau$ such that for all closed terms N_1, N_2 in σ

$$N_1 =_{\beta\eta} N_2 \Leftrightarrow MN_1 =_{\beta\eta} MN_2.$$

σ and τ are *equivalent* if each is reducible to the other.

We denote the equivalence class of σ by $[\sigma]$ and define

$$[\sigma] < [\tau] \qquad \text{iff} \quad \sigma \leqslant \tau \text{ but not } \tau \leqslant \sigma.$$

$$\mu \equiv (((0 \to 0) \to 0) \to 0) \to 0 \to 0, \qquad \nu \equiv (0 \to 0) \to (0 \to 0) \to 0 \to 0.$$

In [Statman, 1980] the following theorem is proved:

* One of the referees pointed out that a solution for the problem was also found, but not published, by M. Zaionc of the University of Krakow, Poland.

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STATMAN'S THEOREM. *The equivalence classes of types are well ordered in type $\omega+2$ or $\omega+3$. A system of representatives is the following*:

$$\begin{array}{ll}
0. & 0 \\
1. & 0 \to 0 \\
\cdots\cdots\cdots & \\
n. & \underbrace{0 \to (0 \to \cdots (0}_{n \text{ times}} \to 0) \cdots) \\
\cdots\cdots\cdots & \\
\omega. & (0 \to 0) \to 0 \to 0 \\
\omega+1. & (0 \to 0) \to (0 \to 0) \to 0 \to 0 \equiv \nu \\
\omega+2\,(?) & (((0 \to 0) \to 0) \to 0) \to 0 \to 0 \equiv \mu \\
\omega+3. & (0 \to (0 \to 0)) \to 0 \to 0.
\end{array}$$ ∎

Note that 0 represents the types with no closed terms. After $\omega+2$ there is a question mark in Statman's theorem because it is not clear whether the reducibility "$\nu \leqslant \mu$" is strict. We shall show that indeed $\nu < \mu$, i.e., not $\mu \leqslant \nu$. As a consequence, the question mark may be omitted and we can conclude that the types are well ordered in type $\omega+3$.

THEOREM 2. *μ is not reducible to ν.*

2. THE PROOF OF THE THEOREM

We start with some notations and definitions. Then in Lemmas 7 and 8 we determine the syntactic form of closed terms of type $\mu \to \nu$. Lemma 9 is a technical but central lemma in the proof. From these three lemmas we deduce Proposition 12 by a rather simple induction argument and the theorem follows as Corollary 13.

Notation 3. (i) $1 \equiv 0 \to 0$; $2 \equiv (0 \to 0) \to 0$.

(ii) u is a fixed variable of type μ.

(iii) $\mathscr{V}$ is a fixed collection of variables of types 0, 1, and 2, infinitely many variables of each type:

$n, n', n_1, n_2 \cdots$	range over variables of type 0 in $\mathscr{V}$,
$f, f', f_1, f_2 \cdots$	range over variables of type 1 in $\mathscr{V}$, and
$F, F', F_1, F_2 \cdots$	range over variables of type 2 in $\mathscr{V}$.

(iv) $g, g', g_1, g_2 \cdots$ also denote variables of type 1 (either or not in $\mathscr{V}$).

(v) $\mathscr{P}$ is the set of all terms of type 1 without free variables of type 0.

(vi) $M \in \sigma$ denotes "M is a term of type σ."

DEFINITION 4. $M \in \Lambda^{\tau}$ is in *long $\beta\eta$-normal form* iff

$$M \equiv \lambda x_1 \cdots x_n \cdot x M_1 \cdots M_m,$$

where $xM_1 \cdots M_m$ has type 0 and each M_i is in long $\beta\eta$-normal form. (Note that each $M \in \Lambda^{\tau}$ has a (unique) long $\beta\eta$-normal form.)

EXAMPLE 5. Let x be a variable of type $(0 \to 0) \to ((0 \to 0) \to 0) \to 0$. Then x is in $\beta\eta$-normal form, but x is not in long $\beta\eta$-normal form. $x =_{\beta\eta} \lambda fF \cdot xfF =_{\beta\eta} \lambda fF \cdot x(\lambda n \cdot fn)(\lambda f' \cdot Ff') =_{\beta\eta} \lambda fF \cdot x(\lambda n \cdot fn)(\lambda f' \cdot F(\lambda n' \cdot f'n'))$ and this last term is the long $\beta\eta$-normal form of x.

DEFINITION 6. $\mathscr{A}$ is the collection of terms defined by:

(1) $n \in \mathscr{V} \Rightarrow n \in \mathscr{A}$;

(2) $M \in \mathscr{A}, f \in \mathscr{V} \Rightarrow fM \in \mathscr{A}$;

(3) $M \in \mathscr{A},\ F, n \in \mathscr{V} \Rightarrow F(\lambda n \cdot M) \in \mathscr{A}$.

$\mathscr{B}$ is the collection of terms defined by:

(1) $n \in \mathscr{V} \Rightarrow n \in \mathscr{B}$;

(2) $M \in \mathscr{B}, f \in \mathscr{V} \Rightarrow fM \in \mathscr{B}$;

(3) $M \in \mathscr{B}, F, n \in \mathscr{V} \Rightarrow F(\lambda n \cdot M) \in \mathscr{B}$;

(4) $M_1, M_2 \in \mathscr{B}, F \in \mathscr{V} \Rightarrow u(\lambda F \cdot M_1)\, M_2 \in \mathscr{B}$.

LEMMA 7. *Let $M \in 0$ be in long $\beta\eta$-normal form. Then*

a. $FV(M) \subset \mathscr{V} \Leftrightarrow M \in \mathscr{A}$;

b. $FV(M) \subset (\mathscr{V} \cup \{u\}) \Leftrightarrow M \in \mathscr{B}$.

Proof. $(\Leftarrow)$ Trivial in both cases.
$(\Rightarrow)$ By a simple induction on the length of M. We only give the proof for b: Let $M \in 0$ be in long $\beta\eta$-normal form with $FV(M) \subset (\mathscr{V} \cup \{u\})$. Then there are 4 possibilities

(1) $M \equiv n$

(2) $M \equiv fM_1$

(3) $M \equiv FN^{0 \to 0} \equiv F(\lambda n \cdot M_1)$

(4) $M \equiv uN^{((0 \to 0) \to 0) \to 0} M_2 \equiv u(\lambda F \cdot M_1)\, M_2$

with $M_i \in 0$, M_i in long $\beta\eta$-normal form, and $FV(M_i) \subset (\mathscr{V} \cup \{u\})$ for $i = 1, 2$.

In case (1) the result is immediate. In cases (2)–(4) we have $M_i \in \mathscr{B}$ by the induction hypothesis, so $M \in \mathscr{B}$. ∎

LEMMA 8. *Let $U \in \mu \to \nu$, U closed. Then for some $L \in \mathscr{B}$,*

$$U =_{\beta\eta} \lambda u f_1 f_2 n \cdot L.$$

Proof. $U =_{\beta\eta} \lambda u f_1 f_2 n \cdot L$ with $L \in 0$ and $FV(L) \subset \{u, f_1, f_2, n\} \subset (\mathscr{V} \cup \{u\})$. We may choose L in long $\beta\eta$-normal form and the assertion follows directly from Lemma 7. ∎

LEMMA 9. *Let $L \in \mathscr{A}$. Then one of the following two cases holds:*

(i) *There exist $P \in \mathscr{P}$ and $n \in \mathscr{V}$ such that for each $H \in 2$,*

$$(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = Pn.$$

(ii) *There exist $P_1 \in \mathscr{P}$ and $P_2 \in \mathscr{P}$ such that for each $H \in 2$,*

$$(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = P_1(HP_2).$$

(In this case g should be taken outside $FV(H)$.)

Proof. By induction on the generation of L in $\mathscr{A}$:

$L \equiv n$. Then $(\lambda F \cdot L)(\lambda g \cdot g(Hg)) = n$, so (i) holds with $P \equiv \lambda n' \cdot n'$.

$L \equiv fL'$. We distinguish two cases for L':

Case (i) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = Pn$ for some $P \in \mathscr{P}$ and each $H \in 2$. Then $(\lambda F \cdot fL')(\lambda g \cdot g(Hg)) = f(Pn) = (\lambda n' \cdot f(Pn'))\, n$ with $\lambda n' \cdot f(Pn') \in \mathscr{P}$.

Case (ii) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = P_1(HP_2)$ for each $H \in 2$. Then $(\lambda F \cdot fL')(\lambda g \cdot g(Hg)) = f(P_1(HP_2)) = (\lambda n' \cdot f(P_1 n'))(HP_2)$.

$L \equiv F'(\lambda n' \cdot L')$. We may suppose $n' \notin FV(H)$. Again we distinguish two cases for L':

Case (i) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = Pn$ for each $H \in 2$.

(a) Suppose $F' \not\equiv F$. Then

$$\begin{aligned}(\lambda F \cdot L)(\lambda g \cdot g(Hg)) &= (\lambda F \cdot F'(\lambda n' \cdot L'))(\lambda g \cdot g(Hg)) \\ &= F'(\lambda n' \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg))) \\ &= F'(\lambda n' \cdot Pn) = (\lambda n'' \cdot F'(\lambda n' \cdot Pn''))\, n.\end{aligned}$$

(b) Suppose $F' \equiv F$.

(b1) Suppose $n' \not\equiv n$. Then

$$\begin{aligned}(\lambda F \cdot L)(\lambda g \cdot g(Hg)) &= (\lambda F \cdot F(\lambda n' \cdot L'))(\lambda g \cdot g(Hg)) \\ &= (\lambda g \cdot g(Hg))(\lambda n' \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg))) \\ &= (\lambda g \cdot g(Hg))(\lambda n' \cdot Pn) = Pn.\end{aligned}$$

(b2) Suppose $n' \equiv n$. We show that (ii) in the lemma holds for L. Let $H \in 2$. Then

$$\begin{aligned}(\lambda F \cdot L)(\lambda g \cdot g(Hg)) &= (\lambda F \cdot F(\lambda n \cdot L'))(\lambda g \cdot g(Hg)) \\ &= (\lambda g \cdot g(Hg))(\lambda n \cdot (\lambda F \cdot L')(\lambda g \cdot g(Hg))) \\ &= (\lambda g \cdot g(Hg))(\lambda n \cdot Pn) \\ &= (\lambda n \cdot Pn)(H(\lambda n \cdot Pn)) = P(HP).\end{aligned}$$

Case (ii) $(\lambda F \cdot L')(\lambda g \cdot g(Hg)) = P_1(HP_2)$ for each $H \in 2$.

(a) $F' \not\equiv F$. This case is trivial again.

(b) $F' \equiv F$. Then

$$\begin{aligned}(\lambda F \cdot L)(\lambda g \cdot g(Hg)) &= (\lambda F \cdot F(\lambda n' \cdot L'))(\lambda g \cdot g(Hg)) \\ &= (\lambda g \cdot g(Hg))(\lambda n' \cdot P_1(HP_2)) = P_1(HP_2). \quad \blacksquare\end{aligned}$$

Notation-Definition 10. $y_1 \in ((0 \to 0) \to 0) \to 0$ and $y_2 \in 0$ are variables.

$$M_i = \lambda y_1 y_2 \cdot y_1(\lambda g_1 \cdot g_1(y_1(\lambda g_2 \cdot g_2(g_i y_2)))) \in \mu \qquad \text{for} \quad i = 1, 2.$$

(Note that M_i is a closed term.)

We are going to prove that for each closed $U \in \mu \to \nu$ the term UM_i does not depend on i (modulo $\beta\eta$-conversion). We start with a lemma on M_i.

LEMMA 11. *For each $L_1, L_2 \in \mathscr{A}$ and $F \in \mathscr{V}$ the term $M_i(\lambda F \cdot L_1) L_2$ does not depend on i (modulo $\beta\eta$-conversion).*

Proof. Let $G_i = \lambda f_1 f_2 n \cdot f_i n$ for $i = 1, 2$. Then

$$M_i = \lambda y_1 y_2 \cdot y_1(\lambda g_1 \cdot g_1(y_1(\lambda g_2 \cdot g_2(G_i g_1 g_2 y_2))))$$

and

$$M_i(\lambda F \cdot L_1) L_2 = (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_i g_1 g_2 L_2)))).$$

We apply Lemma 9 to $L = L_1$.

Case 1. 9(i) holds for L_1: There exist $P \in \mathscr{P}$ and $n \in \mathscr{V}$ such that for each $H \in 2$

$$(\lambda F \cdot L_1)(\lambda g \cdot g(Hg)) = Pn.$$

Take $g = g_1$ and $H = \lambda g' \cdot (\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_i g' g_2 L_2))$. Then it follows that $M_i(\lambda F \cdot L_1)\, L_2 = Pn$ for $i = 1, 2$.

Case 2. 9(ii) holds for L_1: There exist $P_1 \in \mathscr{P}$ and $P_2 \in \mathscr{P}$ such that for each $H \in 2$,

$$(\lambda F \cdot L_1)(\lambda g \cdot g(Hg)) = P_1(HP_2).$$

Let $H_{1i} = \lambda g' \cdot P_1(G_i g' P_2 L_2)$ and $H_{2i} = \lambda g' \cdot G_i g_1 g' L_2$, where g_1 is already bound in M_i.

We may assume that $g_j \notin FV(H_{ji})$ for $j = 1, 2$ (if necessary, replace g_j in the definition of M_i by a fresh variable). Now

$$\begin{aligned} M_i(\lambda F \cdot L_1)\, L_2 &= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(G_i g_1 g_2 L_2)))) \\ &= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1((\lambda F \cdot L_1)(\lambda g_2 \cdot g_2(H_{2i} g_2)))) \\ &= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(P_1(H_{2i} P_2))) \\ &= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(P_1(G_i g_1 P_2 L_2))) \\ &= (\lambda F \cdot L_1)(\lambda g_1 \cdot g_1(H_{1i} g_1)) = P_1(H_{1i} P_2) \\ &= P_1(P_1(G_i P_2 P_2 L_2)) \\ &= P_1(P_1(P_2 L_2)). \quad \blacksquare \end{aligned}$$

PROPOSITION 12. *For each closed* $U \in \mu \to \nu$ *the term* UM_i *does not depend on* i *(modulo* $\beta\eta$*-conversion).*

COROLLARY 13. *A closed term* $U \in \mu \to \nu$ *cannot be injective (for closed terms* w.r.t. $\beta\eta$*-conversion). In particular, not* $\mu \leqslant \nu$.

Proof. Immediate. ∎

Proof of Proposition 12. By Lemma 8 we have $U = \lambda u f_1 f_2 n \cdot L$ for some $L \in \mathscr{B}$. Then $UM_i = \lambda f_1 f_2 n \cdot L[u := M_i]$.

We show *by induction on the generation of* L *in* $\mathscr{B}$ that $L[u := M_i]$ does not depend on i:

$L \equiv n'$. This case is trivial.

$L \equiv fL'$. $L'[u := M_i]$ does not depend on i (induction hypothesis) so $(fL')[u := M_i] = fL'[u := M_i]$ does not depend on i.

$L \equiv F(\lambda n' \cdot L')$. This case is also trivial because M_i has no free variables, so

$$(F(\lambda n' \cdot L'))[u := M_i] = F(\lambda n' \cdot L'[u := M_i]).$$

$L \equiv u(\lambda F \cdot L_1)\, L_2$, with $L_1, L_2 \in \mathscr{B}$. Now $L[u := M_i] = M_i(\lambda F \cdot L_1[u := M_i])(L_2[u := M_i])$, where $L_1[u := M_i]$ and $L_2[u := M_i]$ do not depend on i (induction hypothesis). Moreover, $L_j[u := M_i] \in 0$ and $FV(L_j[u := M_i]) \in \mathscr{V}$ for $j = 1, 2$. So the long $\beta\eta$-normal form of $L_j[u := M_i]$ is in $\mathscr{A}$ (by Lemma 7). Now it follows from Lemma 11 that $L[u := M_i]$ does not depend on i. ∎

Remark 14. Proposition 12, and its proof, remain valid if, for a fixed $k \geqslant 0$, we replace i, g_i, and M_i by

$$\mathbf{i} = (i_1, i_2, ..., i_k) \qquad \text{with} \quad i_1, i_2, ..., i_k \in \{1, 2\},$$
$$g_{\mathbf{i}} = \lambda n' \cdot g_{i_1}(g_{i_2}(\cdots (g_{i_k}(n') \cdots),$$
$$M_{\mathbf{i}} = \lambda y_1\, y_2 \cdot y_1(\lambda g_1 \cdot g_1(y_1(\lambda g_2 \cdot g_2(g_{\mathbf{i}}\, y_2)))).$$

The definitions of $G_{\mathbf{i}}$, $H_{1\mathbf{i}}$ and $H_{2\mathbf{i}}$ in the proof of Lemma 11 are obvious. At the end we get $P_1(P_1(P_2 \cdots (P_2(L_2) \cdots)$ (k times P_2).

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REFERENCE

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